Optimal Communication Complexity of Chained Index

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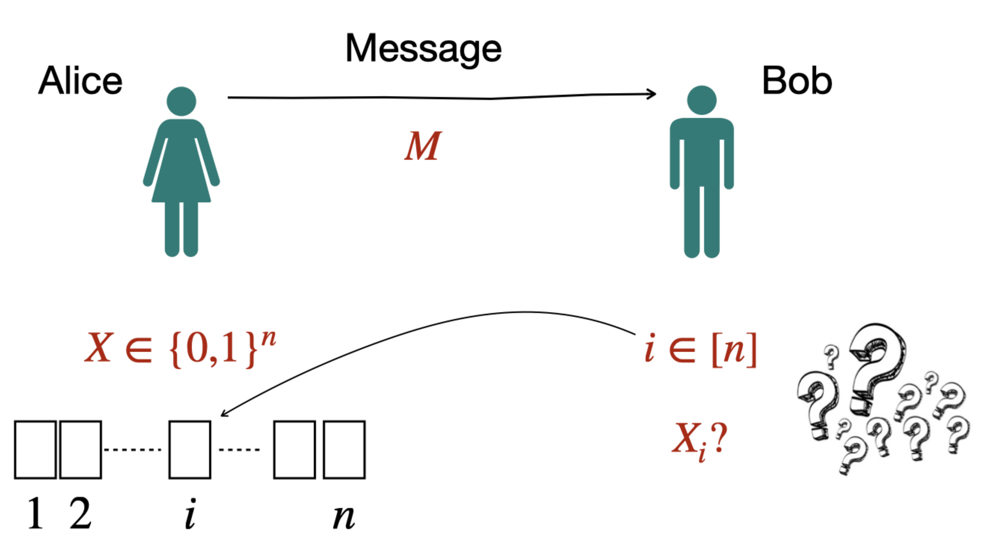
Index Problem

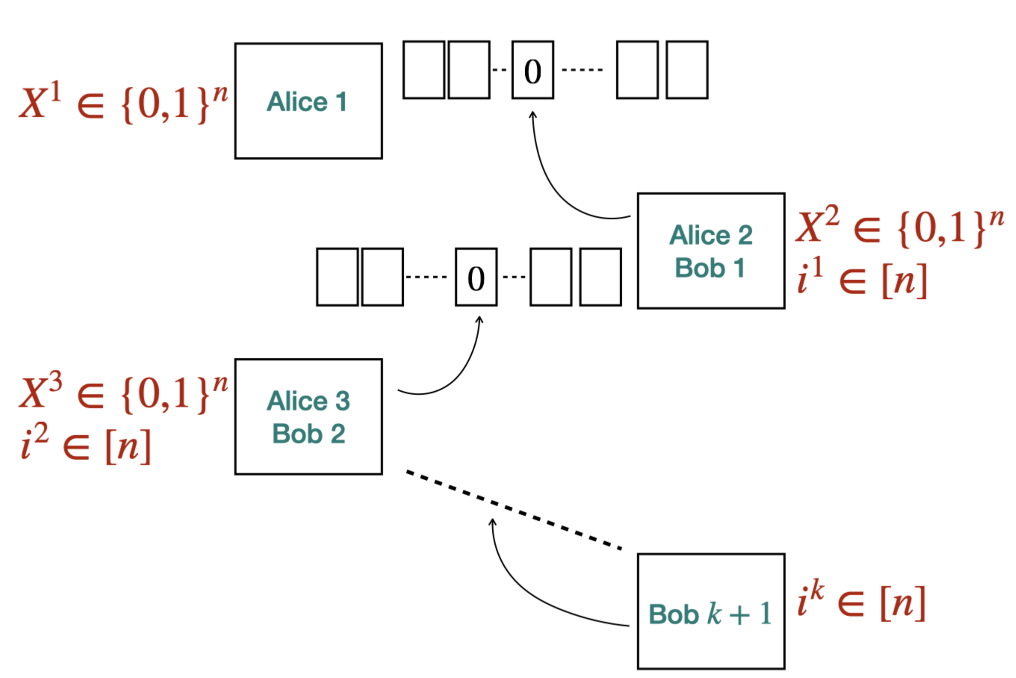
Alice has a string $X \in \{0,1\}^n$ and Bob holds an index $i \in [n]$. Alice sends a message M to Bob who wants to know what X_i is.



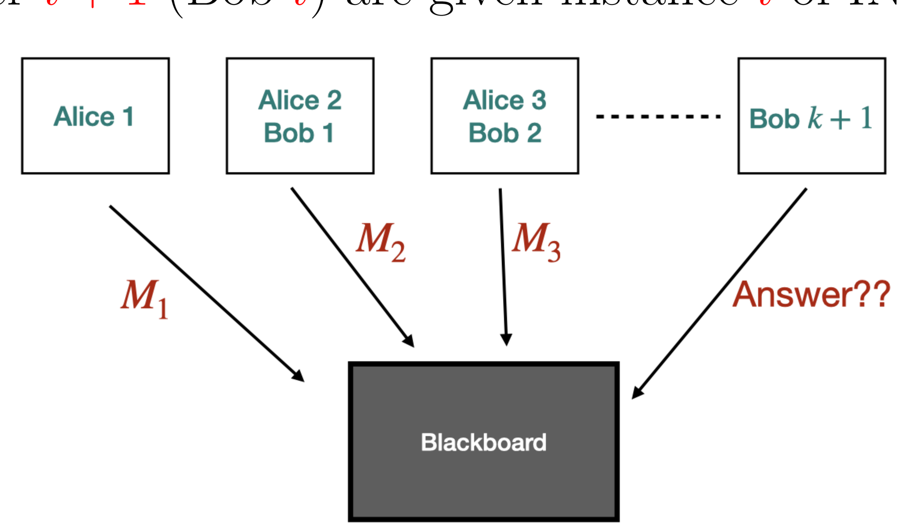
Any protocol π for INDEX needs $\Omega(n)$ bits [1].

Our Problem - Chained Index

Many instances with **same answer** are "chained" together [2].



There are $k+1$ players, and player i (Alice i) and player $i+1$ (Bob i) are given instance i of INDEX.



Messages are sent in order of $M_1, M_2, \ldots, M_k$.
What is the total length of messages on the board?

Our Result

Communication Complexity of Chained Index is $\Omega(n - k\log n)$. Our lower bound is tight, barring corner case when $k = \Omega(n/\log n)$.

Prior Work

Work	Lower bound
[2]	$\Omega(n/k^2)$
[3]	One message with $\Omega(n/k^2)$
[4]	$\Omega(n/k + \sqrt{n})$

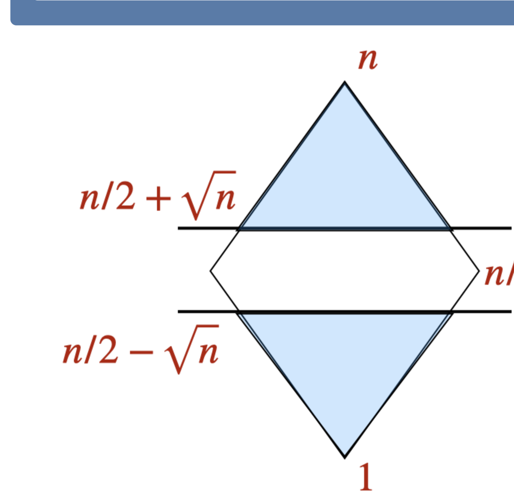
Applications in streaming lower bounds for maximum independent sets [2], streaming submodular maximization [3], and interval selection.

Protocols for INDEX

Assume X, i chosen uniformly at random. How to succeed with probability $1/2 + \Omega(1/\sqrt{n})$?

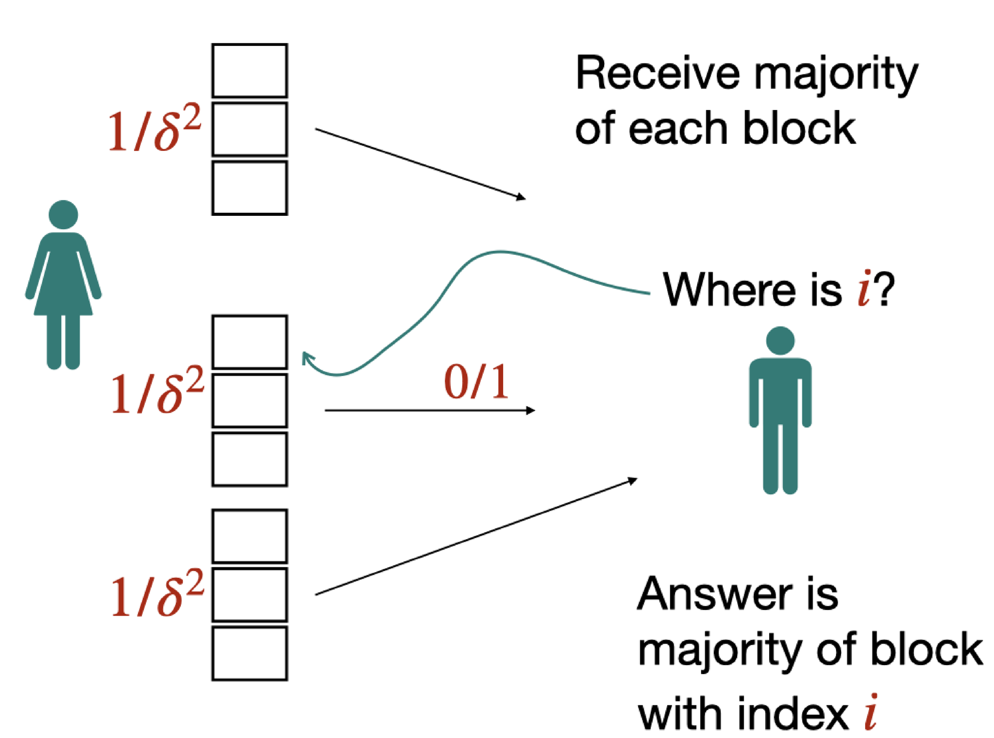
One Bit Protocol

Gain $\Omega(1/\sqrt{n})$ advantage over random guessing by sending majority bit.



Constant fraction of input in blue region. Can be extended to $n\delta^2$ **bits** and $1/2 + \Omega(\delta)$ success probability.

Setting $\delta = 1/k$?



An $O(n/k)$ protocol?

- Each player sends $O(n/k^2)$ bits, forming majority of blocks of size k^2.
- Final player chooses majority of all the answers.

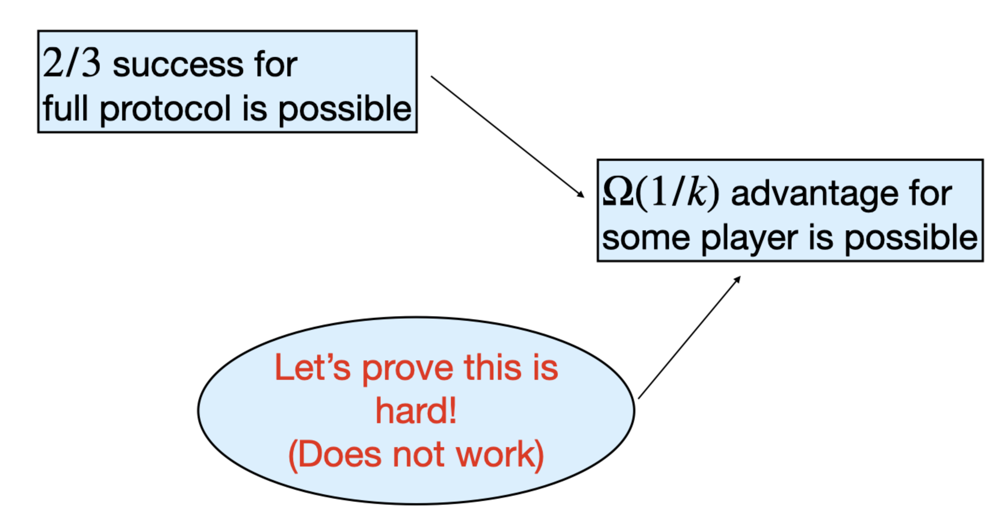
For correctness of answers:

Mean : $k/2 + \Omega(1)$

Standard Deviation : $\Omega(\sqrt{k})$.

Need $\delta = \Omega(1/\sqrt{k})$!

Standard Hybrid Argument

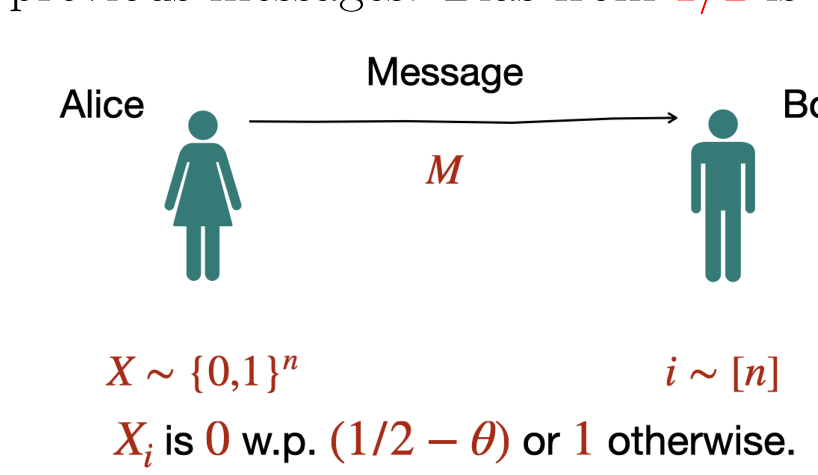


Takeaway

**Some other way to measure progress instead of advantage :
Keep track of Information/Entropy of answer directly.**

Biased Index Problem

Bob already knows something about the answer from previous messages. Bias from $1/2$ is θ.



Results for Biased Index

To find the answer with high probability, entropy should be low.

Initial Entropy

Index	$H_2(1/2) = 1$
Biased Index	$H_2(1/2+\theta) < 1$

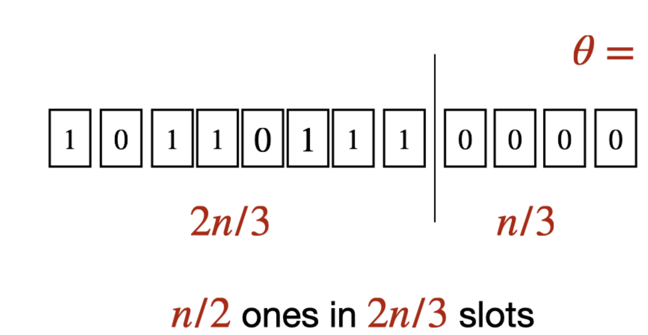
Entropy after Message of Length s

Index	$1 - O(s/n)$
Biased Index	$H_2(1/2+\theta) - O((s+\log n)/n)$

Another Sampling Process [5]

- Restrict to X with exactly $n/2$ ones. (Loss of $\log n$ in entropy.)
- Sample set $T \subset [n]$ of size $n/(1+2\theta)$ uniformly.
- Sample $n/2$ positions from T, setting them to 1. Rest of the positions are set to 0.
- Choose i uniformly random from T.

Conditioned on set T, string X and index i are independent!



One Application

$\Omega(n^2/\alpha^5 - \log n)$ lower bound for α-approximation of maximum independent sets in vertex arrival streams.

References

[1] I. Kremer, N. Nisan, D. Ron, STOC '95, 596-605.

[2] G. Cormode, J. Dark, C. Konrad, ICALP '19, 45:1-45:14.

[3] M. Feldman, A. Norouzi-Fard, O. Svensson, R. Zenklusen, STOC '20, 1363-1374

[4] M. Huang, X. Mao, G. Yang, J. Zhang, ECCC, TR24-067.

[5] S. Assadi, S. Khanna, Y. Li, STOC '16, 698-711.