Multicriteria Spanners – A New Tool for Network Design

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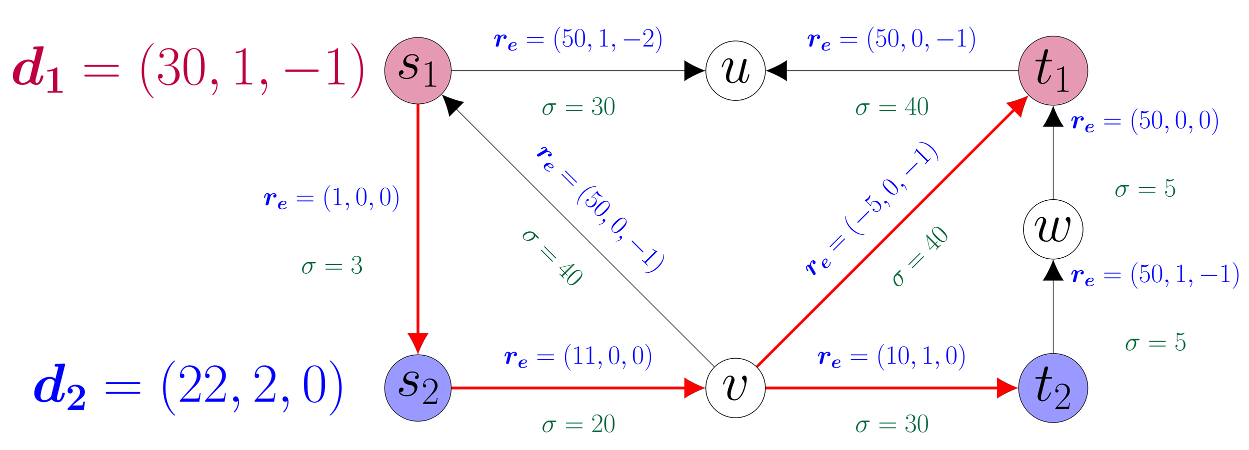
Multicriteria Spanners



- Each edge e has an $m+1$ dimensional resource consumption vector $\boldsymbol{r_e} \in \mathbb{R} \times [\tau]_{\pm}^{m}$: where $[\tau]_{\pm}^{m} = \{-\tau, -\tau+1, \ldots, -1, 0, 1, \ldots, \tau\}$ (m is a constant).
- For a specific resource $i \in [m+1]$, we assume that for every $e \in E$, either $\boldsymbol{r_e}[i] \geq 0$ (these resources are called covering resources) or $\boldsymbol{r_e}[i] \leq 0$ (these are packing resources).
- Edges have an upfront/investment cost $\sigma(e) \in \mathbb{Q}_{\geq 0}$.
- If $m = 0$, the problem reduces to Weighted Spanners.

Goal: pick a subgraph H so that

- the total cost : $\sum_{e \text{ is picked}} \sigma_e +$ is minimized and
- there is an $s_i \rightsquigarrow t_i$ path in H with resource consumption not more than the budget.

Some applications of Multi criteria spanners

Packing resources:

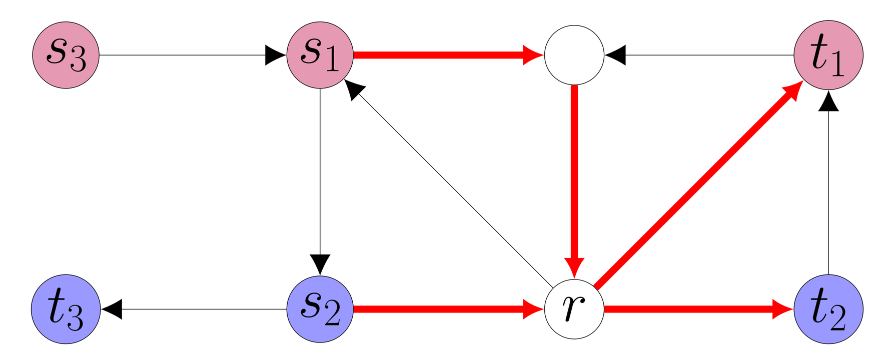
- Some problems in Hop-constrained network design can be captured by Multi criteria spanners We can use one of the resources to encode the hop constraint.
- Edge avoiding spanners where individual demand pairs may need to avoid certain edges (e.g., a heavy truck needs to avoid bridges).

Covering resources:

- The spanner extension of Group Assymetric travelling salesman problem (ATSP) . In Group ATSP, we are given a weighted directed graph $G = (V, E)$ and a collection of subsets $R_1, \ldots, R_k$ of V. The goal is to find a minimum-cost tour that visits at least one vertex in each R_i. For the spanner extension, we can use one covering resource for each group and tweak the budget to ensure the required groups are visited.

Multicriteria spanners can also handle combinations of these applications by adding extra resources!

Main Tool: Junction trees



$$(\text{density}(J) = \frac{\text{cost of edges in} J}{\text{number of satisfied pairs in} J})$$

Junction trees are low-weight structures that connect a lot of demand through paths that go through the same vertex. They are useful as tools in several network design algorithms.

Our work extends junction trees for the MULTI CRITERIA SPANNER setting (where we have vector constraints). These new junction trees are called MIN-DENSITY RESOURCE CONSTRAINED JUNCTION TREE (RCJT).

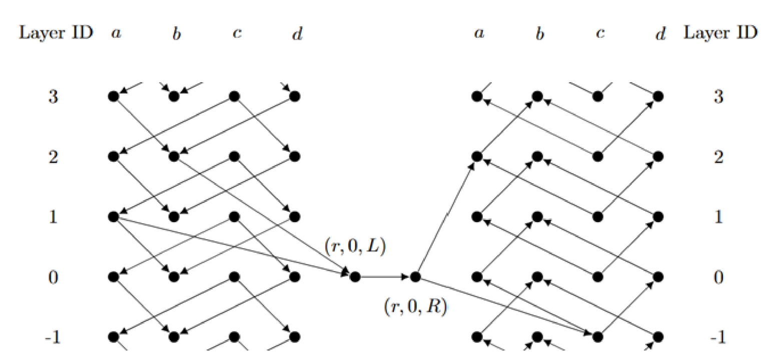
Theorem (Min-density junction tree)

For any constant $\varepsilon > 0$, there is a polynomial-time (under mild assumptions) algorithm that gives an $\tilde{O}(k^{\varepsilon} \cdot \tau^{m})$-approximation for MIN-DENSITY RCJT.

Main subtool for Min-density RCJT: Group Steiner Tree

- **Group Steiner Tree (GST):** Given a weighted undirected graph $G = (V, E)$, as well as k subsets $R_1, \ldots, R_k$ of V , find a minimum-cost subtree T of G containing at least one vertex from each R_i.
- **GST** has an LP formulation with a small integrality gap when the underlying graph is rooted tree with a small height h.
- **Conclusion:** If we can turn the MIN-DENSITY RCJT into a shallow tree input of GST, then we have a good solution for MIN-DENSITY RCJT.
- This requires us to
  1. turn the resource constraints into connectivity constraints with a new graph $\hat{G}$.
  2. turn the graph $\hat{G}$ into a shallow tree $\hat{T}$.

The main algorithm: (based on [CDKL'20, CEGS' 11])



- Build a large product graph where every vertex is of form (VERTEX ID, LAYER ID); this graph has multiple copies of original vertices and edges. Resource constraints in the original graph can be tracked by connectivity constraints in the product graph.
- When we have $m > 0$, the solution may use multiple copies of the same edge in the product graph. This increases the upper bound on the approximation factor.

Key Observation

Through careful analysis, the number of copies we use can instead be bounded by the number of layers corresponding to just m dimensions . This means that the first dimension does not play a role in approximation factor and we can capture regular spanners!

- The product graph is turned into a shallow tree by means of a blackbox result from [Zelikovsky' 97].
- Build a linear program to solve the connectivity problem on the shallow tree.
- Before rounding, we ensure that whatever terminal we choose to round will satisfy the resource constraints with a multi-round pruning procedure. Then, run the group steiner rounding blackbox.

Theorem (Min-density junction tree)

If $\boldsymbol{r_e}[1] \in \{1, 2, \ldots, T\}$ for all $e \in E$, for any constant $\varepsilon > 0$, there is a polynomial-time (under mild assumptions) algorithm that gives an $\tilde{O}(k^{1/2+\varepsilon} \cdot \tau^{m})$-approximation for MULTICRITERIA SPANNERS running in time $\mathsf{poly}(n, \tau^{m}, T)$.

Overall: Extend Spanners to the multicriteria setting and give approx algorithm with same approx ratio as directed spanners!